Id: 1296285

Participatory Educational Research (PER)
Vol.10(6), pp. 41-57, November 2023
Available online at http://www.perjournal.com
ISSN: 2148-6123
http://dx.doi.org/10.17275/per.23.88.10.6

# An Examination of Mathematics Teachers' Mathematical Language Usage

Zeynep ÖZAYDIN *
*Department of Math Education, Bursa Uludag University, Bursa, Turkey*
*ORCID: 0000-0003-1768-3963*

Menekşe Seden TAPAN BROUTIN
*Department of Math Education, Bursa Uludag University, Bursa, Turkey*
*ORCID: 0000-0002-1860-852X*

Çiğdem ARSLAN
*Department of Math Education, Bursa Uludag University, Bursa, Turkey*
*ORCID: 0000-0001-7354-8155*

**Article history**
**Received:**
12.05.2023

**Received in revised form:**
18.07.2023

**Accepted:**
04.09.2023

**Key words:**
mathematical language, mathematics teachers, three worlds of mathematics

The aim of this research is to examine the mathematical language use of mathematics teachers in detail within the framework of the theoretical approach of "Three Worlds of Mathematics" put forward by Tall (2007). The research was carried out in the case study pattern, which is one of the qualitative research methods. The study group of the research consists of 27 math teachers. In the research, "Worksheet for the Use of Mathematical Language", which includes four mathematical situations, was used as a data collection tool. In the analysis of the data collected within the scope of the research, the interpretation of the "Three Worlds of Mathematics" theoretical approach towards the use of mathematical language was adopted. As a result of the research, it was seen that mathematics teachers used mathematical language mostly in the context of the "conceptual-embodied world", and least in the context of the "axiomatic-formal world". At the same time, while the mathematical world in which the mathematical situation is presented affects the mathematical language used by the teachers for that situation, the teachers could not fully reflect the mathematical language to different worlds whilst working on mathematical situations. Finally, based on the conclusion that teachers use mathematical language in a more qualified way while performing high-level skills, it is suggested to include activities that allow the development of high-level skills in mathematics lessons and that serve to understand and develop mathematics

## Introduction

Mathematics is a universal language based on symbols and shapes (Ministry of National Education [MoNE], 2009). The reason for the recognition of mathematical language

* Correspondency: zeynepozaydin@uludag.edu.tr

as an international language is its unique systematic structure (lexical, symbolic, and iconic) (Uğurel & Moralı, 2010). It can be said that the proper use of this systematic structure with correct content plays a significant role in the formation of mathematical language in individuals. Due to the significant role of mathematics in understanding, it is necessary to develop individuals' skills in the mathematical language (Gawned, 1990). Considering the importance of mathematical language in understanding mathematics, it is necessary to examine how it is addressed in mathematics education curricula.

MoNE (2018), includes the ability for students to use mathematical terminology and language correctly as one of the special objectives of mathematics education in the Mathematics Curriculum. The Australian Mathematics Teaching Curriculum (ACARA, 2022) emphasizes that mathematics focuses on developing the skills and understanding necessary for students to think, reason, and generate solutions to problems by using appropriate mathematical language in given situations. Similarly, the National Council of Teachers of Mathematics (NCTM, 2000) has established the proper use of mathematical language as a standard and included learning mathematical language as one of the fundamental objectives of mathematics education. Therefore, the development of mathematical language is an important target to be achieved in mathematics education curricula.

In addition to the significant place of mathematical language in mathematics education curricula, many educators point out that mathematical language responds to needs such as developing mathematical thinking, mastering mathematical subject knowledge, applying mathematical studies, and practicing mathematical communication (Purpura & Reid, 2016; Hornburg et al., 2018). Mathematical language plays a critical role in helping individuals develop new ways of learning mathematical concepts (Chard, 2003). At the same time, students' use of mathematical language predicts their mathematical skills (McClelland et al., 2007; Purpura & Logan, 2015). Nevertheless, studies in the field show that students have difficulties in using mathematical language (Akarsu, 2013; Güzel & Yılmaz, 2020; Korhonen et al., 2011; Rudd et al., 2008; Yüzerler, 2013).

The main task in creating and implementing a learning environment that allows for the learning and development of mathematical language belongs to the teacher. Mercer and Sams (2006) support this idea by stating the significant role that mathematics teachers play in enabling students to use language effectively. In order for teachers to carry out their profession effectively, they must have a command of the subjects in their field (Shulman, 1986). A teacher who does not have a command of mathematical language cannot be expected to develop their students' mathematical language.

Therefore, in the context of the current research, it is considered important to gain information about the proficiency of mathematics teachers in mathematical language and to examine their use of mathematical language. However, mathematical language is considered a complex language form because it contains abstract terms and concepts with different mathematical meanings (Rubenstein & Thompson, 2002; Schmitt et al., 2019). Due to the complex structure of mathematical language, gaining knowledge about individuals' use of mathematical language can only be achieved through a detailed and in-depth examination.

Tall (2007) has made possible to examine the use of mathematical language in depth and detail within the framework of mathematical thinking, which he called "Three Worlds of Mathematics". According to "Three Worlds of Mathematics", mathematical language develops in conjunction with mathematical thinking (Akarsu Yakar, 2019; Tall, 2008). Tall

(2007) defines three mathematical worlds as (i) "conceptual-embodied world", (ii) "proceptual-symbolic world", and (iii) "axiomatic-formal world", and points out that mathematical language is used in different ways in these three worlds. The use of mathematical language in terms of theoretical approach in the "Three Worlds of Mathematics" is presented in Figure 1.

| Three Worlds of Mathematics | Definition | Use of Mathematical Language |
|---|---|---|
| Conceptual-Embodied World | Interacting with objects, teasing out their properties, to begin to describe them | Used to name concepts and define their properties |
| Proceptual-Symbolic World | Symbolizing actions, reflecting on relationships, building connections between thinking processes | Using symbolic expressions, formulate problems, to talk about what to do |
| Axiomatic-Formal World | Develops theorems from definitions and formal mathematical proof | The terms and theorems are defined to have specific theoretic properties |

Figure 1. Mathematical language uses in terms of "Three Worlds of Mathematics" theoretical approach (Tall, 2007; 2008)

There are various studies in the literature on the use of mathematical language. However, these studies are mostly conducted with pre-service mathematics teachers (Aydın & Yeşilyurt, 2007; Çakmak et al., 2014; Çalıkoğlu Bali, 2002; Emre et al., 2017; Gültekin & Es, 2018; Gürefe, 2018; Kıymaz et al., 2020; Shockey & Pindiprolu, 2015; Yardımcı, 2019; Yeşildere, 2007) and students (Akarsu, 2013; Akarsu Yakar, 2019; Aydoğan Belen, 2018; Güzel & Yılmaz, 2020; Oxenford O'Brian et al., 2010; Purpura & Reid, 2016; Rudd et al., 2008; Yakar & Yılmaz, 2017; Yüzerler, 2013), and it has been observed that there are few studies that examine the use of mathematical language by mathematics teachers (Açıl & Zeybek, 2017; Baiduri & Utomo, 2020; Baki and Çelik, 2018; Kuntze et al., 2018). Açıl and Zeybek (2017) examined mathematics teachers' ability to recognize their students' use of mathematical language, Baiduri and Utumoto (2020) examined the competencies of mathematics teachers in acquiring and using mathematical language, Baki and Çelik (2018) analyzed mathematics teachers' mathematical discourse towards mathematical language, and Kuntze et al. (2018) investigated how mathematics teachers support their students in using multiple representations as part of mathematical language. It has been observed that these studies thoroughly examine the use of mathematical language by mathematics teachers in a teaching and learning process, but they are not bound by a theoretical framework for mathematical language.

In this context, it is seen as important and necessary to examine the mathematical language usage of mathematics teachers who are the implementers of mathematics teaching programs that have an important place for mathematical language and play an effective role in developing students' mathematical language usage in detail within a theoretical framework. At the same time, considering that the studies on mathematical language use conducted in our country are mostly carried out with teacher candidates and students, it is thought that the sample type of the current research will also contribute to the literature. Therefore, the purpose of this research is shaped as the examination of mathematics teachers' mathematical language usage within the theoretical framework of "Three Worlds of Mathematics" approach proposed by Tall (2007). In line with this purpose, the research problem that directs the research is determined as "How is the mathematical language usage of mathematics teachers?"

## Method

This research was conducted using the case study design, which is one of the qualitative research methods. Qualitative research is one of the ways of producing knowledge that people have constructed themselves in social structures (Morgan, 1996). Qualitative research aims to interpret phenomena within their own contexts without altering them (Maxwell, 2008) using an interpretive approach (Baltacı, 2019), providing an understanding of human events in their own context (Bogdan & Biklen, 1997). In case study research, which is one of the qualitative research methods, researchers focus on describing a limited research topic in detail as it exists (Hancock & Algozzine, 2006). Case study research provides an opportunity for the researcher to examine, understand, and draw inferences about an event in depth and without intervention (Yin, 2014). This research is suitable for the case study design, which is one of the qualitative research methods, because it aims to examine the mathematical language usage that teachers construct within their own life contexts, without altering them, within the theoretical framework of "Three Worlds of Mathematics" proposed by Tall (2007).

### *Study Group*

This research is conducted with 27 mathematics teachers working in different cities of Turkey. The study group of the research was determined by convenient sampling, which is one of the purposive sampling methods. In purposive sampling, researchers create the study group to meet the predetermined characteristics (Teddlie & Yu, 2007). Purposive sampling, which allows for more qualitative research by selecting information-rich situations, offers greater depth to research while having limited scope (Patton, 2018, p.230). In convenient sampling, the researcher includes individuals who are close to them, easily accessible, and sufficient in number in the study group (Singleton and Straits, 2005). Therefore, in order to create information-rich situations, a sufficient number of mathematics teachers with different educational levels, graduate programs, professional experiences, and working in different types of schools were included in the study group, which the researcher could easily reach.

### *Data Collection Tools*

The "Worksheet for Mathematical Language Use" was used as the data collection tool in the study. There are a total of four mathematical situations in the worksheet. Mathematical situations were arranged by the researchers in accordance with the theoretical approach of "Three Worlds of Mathematics" put forward by Tall (2007) and were also used in data analysis. Care was taken to present each mathematical situation in the data collection tool to address at least one of the three mathematical worlds in the theoretical approach. While determining the worlds of mathematics to which mathematical situations belong, opinions of one associate professor and five expert mathematics teachers were taken. The fact that the opinions expressed by the experts were the same showed that the data collection tool was structurally valid.

The first mathematical situation is presented in the "conceptual-embodied world", the second in the "proceptual-symbolic world", the third in the "axiomatic-formal world", and the fourth in the context of both “conceptual-embodied world” and “proceptual-symbolic world”. The main aim is to see how the use of mathematical language occurs in different mathematical worlds. However, it has been wondered whether or how the use of mathematical language will change if the mathematical situation includes more than one mathematical world. For this reason, the first three mathematical situation are arranged to address one world, and the fourth mathematical situation to address two worlds. The four mathematical situations in the worksheet are presented in Figure 2.

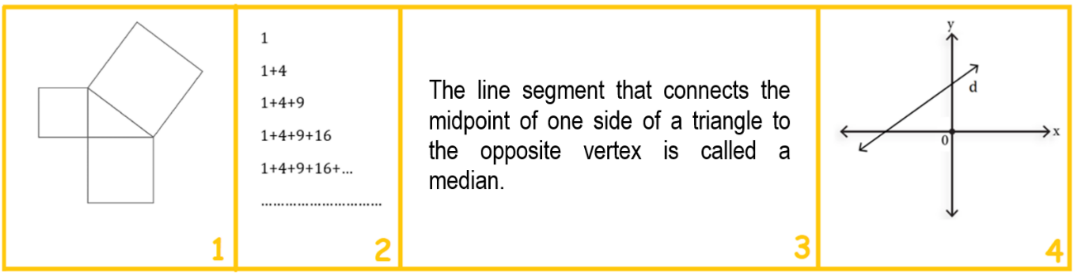



Figure 2. Mathematical situations included in the worksheet for mathematical language use

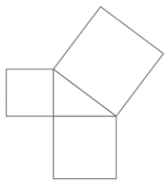
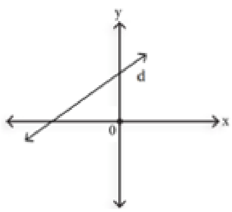
In order to analyze the teachers' usage of mathematical language in detail through mathematical situations, two questions were asked for each mathematical situation. The questions were as follows: (i) What do you see in the given mathematical situation? (ii) Pose and solve a problem using the given mathematical situation. Therefore, a total of eight questions were directed to the mathematics teachers who formed the research group for four mathematical situations. Mathematical problem posing was not considered as a theoretical framework in the related research. During problem posing, which is one of the most effective ways of doing mathematics (NCTM, 2000), individuals develop a mathematical language (Rudnitsky et al., 1995). Based on this understanding, problem posing was used to collect data in this study aiming to examines the use of mathematical language.

### *Data Collection Process and Data Analysis*

The data was collected from mathematics teachers on one occasion for the purpose of the research. The data collection tool was sent to teachers via online platforms, and they were asked to answer the questions and share their answers with the researchers by taking photos. Teachers were informed that their answers would be scientifically evaluated and that their personal information and answers would not be shared with anyone without their permission.

The theoretical approach of “Three Worlds of Mathematics” (Tall, 2007) was adopted to interpret the data collected in the study regarding the use of mathematical language. An analysis framework was prepared for each mathematical situation in accordance with the relevant literature (Tall, 2007; 2008; Akarsu Yakar, 2019, p.49). The prepared analysis framework is presented in Table 1.

Table 1. Analysis framework for each mathematical situation in the data collection tool

| Mathematical Situation | The nature of the expected answer according to the Three World of Mathematics | | |
|---|---|---|---|
| | Conceptual-Embodied World | Proceptual-Symbolic World | Axiomatic-Formal World |
|  | Expressing geometric figures or the Pythagorean theorem in a written or visual form | Expressing Pythagorean theorem in a symbolic form | Defining Pythagorean theorem in a theoretical form |
| 1<br>1+4<br>1+4+9<br>1+4+9+16<br>1+4+9+16+...<br>.............................. | Recognizing the pattern or stating the properties of the pattern | Symbolizing the relationships in the pattern | Defining or proving the formula for the sum of the square numbers |
| The line segment that connects the midpoint of one side of a triangle to the opposite vertex is called a median. | Expressing the mathematical situations contained in the median of a triangle in written or visual form and indicating its properties | Symbolizing the situations created by the median in the triangle | Defining theoretically the median and/or its properties |
|  | Expressing the mathematical situations contained in the given coordinate system in a written or visual form and indicating its properties | Symbolizing the mathematical situations contained in the given coordinate system | Defining the theoretical properties of the equation or the slope of a line |

The responses obtained from the teachers were analyzed according to Table 1. While conducting the analysis, the responses to the question "What do you see in the mathematical situation?" in the data collection tool were considered under the "Seeing/Perceiving" dimension, and the responses to the question "Pose and solve a problem using the given mathematical situation." were taken into account under the "Understanding/Developing" dimension. Since problem posing involves processes such as understanding, designing, creating, problem-solving, and evaluation (Örnek & Soylu, 2021), it was deemed appropriate to name the dimension where the responses to the question "Pose and solve a problem using the given mathematical situation." were analyzed as "Understanding/Developing". The analysis results were tabulated by frequencies and sample teacher response quotations according to the "Three Worlds of Mathematics".

### *Validity and Reliability*

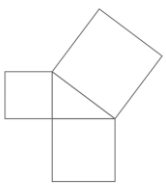
Validity in qualitative research concerns the accuracy of research results (Baltacı, 2019). Detailed analysis and reporting of the obtained data, supporting the researcher's analyses directly with quotations (Yıldırım & Şimşek, 2021), and seeking opinions from different researchers on the findings (Denzin & Lincoln, 2008) are among the important criteria for validity in qualitative research. In the current study, the analysis process of the data has been shared in detail with readers, the analyses have been directly supported with quotations, and the opinions of different researchers have been consulted at every stage of the research, thus increasing the validity.

Reliability, on the other hand, refers to the replicability of the research (Baltacı, 2019). Data analysis that is conducted in adherence to a well-defined theoretical framework before data collection (Baltacı, 2017; Creswell, 2012) and the ability of an external reader to verify research findings by following the analysis method used in the research (Merriam, 1998) are accepted as criteria to ensure reliability. In the current study, the analysis of the data adheres to the theoretical approach of "Three Worlds of Mathematics" proposed by Tall (2007), which increases reliability. Moreover, the analysis framework used in the data analysis includes the expected responses, which supports reliability by enabling an external researcher to follow the analysis method. In this way, of the research can be replicability. Considering all these factors, the current study is concluded to be valid and reliable.

## Results

In this section, the findings obtained within the scope of the research will be discussed in order according to the mathematical situations included in the data collection tool. Based on the findings, the mathematical language use of mathematics teachers will be presented in the context of the “Three Worlds of Mathematics”.

Table 2. The mathematical language use of mathematics teachers in the first mathematical situation presented in the context of “conceptual-embodied world”

| Mathematical Situation Presented in Conceptual-Embodied World Context | Three Worlds of Mathematics | Dimension in which Mathematical Language is Examined | | Total (f) | Examples of expressions |
|---|---|---|---|---|---|
| | | Seeing/ Perceiving (f) | Understanding/ Developing (f) | | |
| | Conceptual-Embodied World | 27 | 27 | 54 | *“Obtain squares using the sides of a right triangle with side lengths of 3, 4, and 5 cm. Then, find a relationship between the areas of these squares.”* |
|  | Proceptual-Symbolic World | 4 | 18 | 22 | *“I see the Pythagorean Theorem.* $a^2+b^2=c^2$*”* |
| | Axiomatic-Formal World | 3 | 5 | 8 | *“I see the Pythagorean Theorem in a right triangle.* $a^2=b^2+c^2$ *a: hypotenuse, b and c: perpendicular sides.”* |

In the first mathematical situation presented in the "conceptual-embodied world" context, 27 responses were obtained for each of the "Seeing/Perceiving" and "Understanding/Developing" dimensions. Of the 27 responses in the "Seeing/Perceiving" dimension, the "conceptual-embodied world" was used in all of them, the "proceptual-symbolic world" in 4, and the "axiomatic-formal world" in 3; in the "Understanding/Developing" dimension, the "conceptual-embodied world" was used in all 27 responses, the "proceptual-symbolic world" in 18, and the "axiomatic-formal world" in 5 related to the mathematical language use.

In his response within the scope of the "Seeing/ Perceiving" dimension, Ö9 included the statement "*I see the Pythagorean Theorem.* $a^2+b^2=c^2$". Ö9 noticed that the geometric shape combination in the mathematical situation represented the Pythagorean Theorem and named it as such. At the same time, Ö9 symbolically expressed the Pythagorean Theorem "$a^2+b^2=c^2$". Therefore, Ö9's statement contains mathematical language in both "conceptual-embodied world" and "proceptual-symbolic world". Since Ö9 did not make any definition for a, b, and c, his statement lacks theoretical properties. Therefore, it does not contain mathematical language in the context of the "axiomatic-formal world". On the other hand, in his response within the scope of the "Seeing/ Perceiving" dimension, Ö14 included the statement "*I see the Pythagorean Theorem in a right triangle.* $a^2+b^2=c^2$, *a: hypotenuse, b and c: perpendicular sides*". By adding an explanation in the form of "$a^2+b^2=c^2$, *a: hypotenuse, b and c: perpendicular sides*," Ö14 utilized a theoretical dimension. Therefore, Ö14's statement was analyzed as containing mathematical language in the context of the "conceptual-embodied world", "proceptual-symbolic world", and "axiomatic-formal world".

Table 3. Mathematical language usages of mathematics teachers in the second mathematical situation presented in the "proceptual-symbolic world" context

| Mathematical Situation Presented in Proceptual-Symbolic World Context | Three Worlds of Mathematics | Dimension in which Mathematical Language is Examined | | Total (f) | Examples of expressions |
|---|---|---|---|---|---|
| | | Seeing/ Perceiving (f) | Understanding/ Developing (f) | | |
| 1<br>1+4<br>1+4+9<br>1+4+9+16<br>1+4+9+16+…<br>………………………… | Conceptual-Embodied World | 27 | 27 | 54 | *"The squares of consecutive numbers are added. A pattern is obtained."* |
| | Proceptual-Symbolic World | 2 | 18 | 20 | *"What is the sum of the numbers that form the number pattern* $1^2+2^2+3^2+4^2+\ldots+30^2$*?"* |

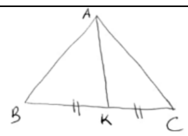
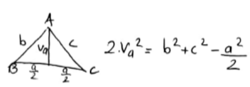
| | | | | | |
|---|---|---|---|---|---|
| | Axiomatic-Formal World | 1 | 5 | 6 | *"How many squares are there in a 4x4 square? Create a relation for an n x n square. For an n x n square, there will be a total of $1^1+2^2+\ldots+(n-1)^2+n^2=n.(n+1).(2n+1)/6$."* |

In the second mathematical situation presented in the "proceptual-symbolic world" context, all 27 responses in the "Seeing/Perceiving" dimension included the "conceptual-embodied world", 2 responses included the "proceptual-symbolic world", and 1 response included the "axiomatic-formal world". In the "Understanding/Developing" dimension, all 27 responses used mathematical language in the "conceptual-embodied world", 18 in the "proceptual-symbolic world", and 5 in the "axiomatic-formal world" context.

Ö16 stated in his response analyzed within the "Seeing/Perceiving" dimension that "*The squares of consecutive numbers are added. A pattern is obtained.*". Ö16 recognized that the pattern in the mathematical situation is obtained by adding the squares of consecutive numbers and expressed this. Ö16's statement was analyzed as containing only mathematical language in the context of the "conceptual-embodied world".

Ö4 included the statement "*What is the sum of the numbers that form the number pattern* $1^2+2^2+3^2+4^2+\ldots+30^2$?" in the problem that he formulated within the scope of the "Understanding/Developing" dimension. Ö4 realized that the numbers in the mathematical situation formed a pattern. At the same time, he expressed with a symbolic language that the pattern was formed by adding the squares of consecutive numbers in the form of "$1^2+2^2+3^2+4^2+\ldots+30^2$". Therefore, Ö4's expression includes mathematical language in both the "conceptual-embodied world" and "proceptual-symbolic world".

Table 4. Mathematical language usage of mathematics teachers in the third mathematical situation presented in a "axiomatic-formal world" context

| Mathematical Situation Presented in Axiomatic-Formal World Context | Three Worlds of Mathematics | Dimension in which Mathematical Language is Examined | | Total (f) | Examples of expressions |
|---|---|---|---|---|---|
| | | Seeing/ Perceiving (f) | Understanding/ Developing (f) | | |
| | Conceptual-Embodied World | 12 | 27 | 39 |  |
| The line segment that connects the midpoint of one side of a triangle to the opposite vertex is called a median. | Proceptual-Symbolic World | 4 | 23 | 27 |  |
| | Axiomatic-Formal World | - | 5 | 5 | *"The intersection point of the medians in a triangle is the centroid."* |

In the third mathematical situation presented in the "axiomatic-formal world" context, in the "Seeing/Perceiving" dimension, 12 out of 27 responses included the mathematical language usage in the "conceptual-embodied world", 4 responses included in the "proceptual-symbolic world"; in the "Understanding/Developing" dimension, all 27 responses included the mathematical language usage in the "conceptual-embodied world", 23 responses included in the "proceptual-symbolic world", and 5 responses included in the "axiomatic-formal world" context. It was observed that teachers did not use mathematical language in the "axiomatic-formal world" context in the 27 responses they gave for the "Seeing/Perceiving" dimension.

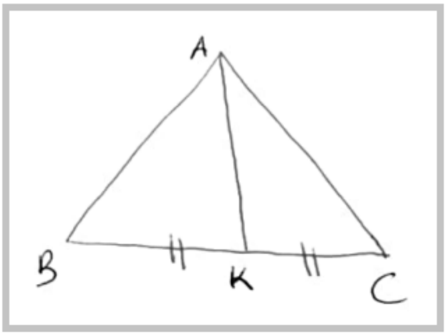



Figure 3. The answer given by Ö7 in the "Seeing/ Perceiving" dimension of the third mathematical situation

As seen in Figure 3, Ö7 has visually expressed the median ([AK]) by drawing a triangle (ABC) in his response analyzed within the scope of the "Seeing/Perceiving" dimension. At the same time, he included a symbolic definition that expresses that the median divides the side [BC] into two equal parts. Therefore, Ö7 used mathematical language in both "conceptual-embodied world" and "proceptual-symbolic world" in his response.

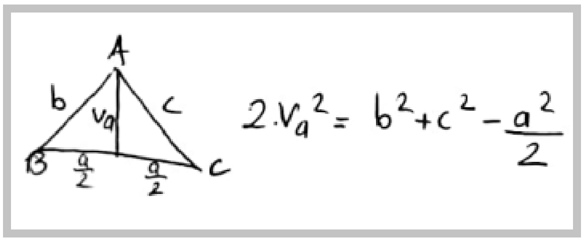
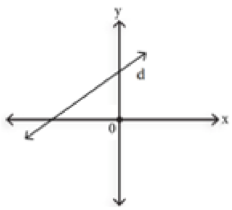



Figure 4. The response of Ö17 in the third mathematical situation within the scope of the "Understanding/Developing" dimension

As seen in Figure 4, Ö17 included the expression "$2V_a^2=b^2+c^2-a^2/2$" while solving his problem in the context of the "Understanding/Developing" dimension. Ö17 symbolically defined the case formed by the median in the triangle. While making this definition, Ö17 drew a triangle and expressed a, b, c, and $V_a$ on the triangle. Therefore, Ö17's symbolic definition is a formal definition with theoretical properties. At the same time, through the triangle he drew, he visually expressed the mathematical situations created by the median in the triangle. Ö17's expression contains mathematical language in both "conceptual-embodied world", "proceptual-symbolic world", and "axiomatic-formal world".

Table 5. Mathematical language uses of mathematics teachers in the fourth mathematical situation presented in "conceptual-embodied world" and "proceptual-symbolic world"

| Mathematical Situation Presented in Conceptual-Embodied and Proceptual-Symbolic World Context | Three Worlds of Mathematics | Dimension in which Mathematical Language is Examined | | Total (f) | Examples of expressions |
|---|---|---|---|---|---|
| | | Seeing/ Perceiving (f) | Understanding/ Developing (f) | | |
| | Conceptual-Embodied World | 27 | 27 | 54 | *"I see the graph of a line that intersects the axes."* |
|   | Proceptual-Symbolic World | 7 | 25 | 32 | *"The general equation of this line that intersects the axes can be written as y=ax+b"* |
| | Axiomatic-Formal World | - | 6 | 6 | *"The slope of a line that leans to the right is positive."* |

In the fourth mathematical situation presented in the context of the both "conceptual-embodied world" and "proceptual-symbolic world", all 27 responses in the "Seeing/Perceiving" dimension involved the "conceptual-embodied world", and 7 of them involved the "proceptual-symbolic world". In the "Understanding/Developing" dimension, all 27 responses involved the "conceptual-embodied world", 25 involved the "proceptual-symbolic world", and 6 involved mathematical language in the "axiomatic-formal world" context. Mathematical language in the "axiomatic-formal world" context was not observed in the responses provided by the teachers in the "Seeing/Perceiving" dimension.

Ö9 included the statement "*The slope of a line that leans to the right is positive*" while solving the problem that he posed in the context of the "Understanding/Developing" dimension. By specifying the property of the line on the coordinate plane in the mathematical situation, Ö9 made a definition with theoretical properties related to the slope. Therefore, Ö9's statement includes mathematical language in both "conceptual-embodied world" and "axiomatic-formal world" contexts.

Ö20 gave the expression "*The general equation of this line that intersects the axes can be written as* $y = ax + b$" in the context of the "Seeing/Perceiving" dimension. Ö20 made a symbolic definition by stating the general equation of the line as "$y = ax + b$" in the expression where he stated the property of intersecting the axes in the mathematical situation. Therefore, the expression of Ö20 contains mathematical language in both "conceptual-embodied world" and "proceptual-symbolic world".

## Conclusion, Discussion and Recommendations

The current study revealed the use of mathematical language by mathematics teachers within the framework of Tall's (2007) "Three Worlds of Mathematics" theoretical approach. As a result of the study, it was observed that mathematics teachers use mathematical language mostly in the context of the "conceptual-embodied world". Çakmak et al. (2014) found that pre-service mathematics teachers had significantly higher verbal language scores than symbolic language scores; and similarly, Emre et al. (2017) determined that teacher candidates preferred to use verbal language especially when trying to conceptualize a concept. The use of mathematical language in the "conceptual-embodied world" is defined related to naming concepts by Tall (2007). Although these studies focused on teacher candidates, they can be considered to have reached similar results to the current study, since verbal language is more frequently employed when naming concepts. Teachers should structure their use of mathematical language with words and symbols that students truly understand while achieving the goals of mathematics education (Jupri & Drijvers, 2016; Mulwa, 2015). The fact that teachers use mathematical language mostly in the context of the "conceptual-embodied world" can be explained by the success of middle school students in performing mathematical language skills in the "conceptual-embodied world" compared to other worlds (Akarsu Yakar, 2019).

This research has also concluded that mathematics teachers use mathematical language the least in the "axiomatic-formal world" context. Similarly, Baki and Çelik (2018) have stated that mathematics teachers use non-formal definitions more when explaining mathematical concepts. However, the formal dimension of mathematical language increases the potential for communication and reduces uncertainty (Harvey, 1982 cited in Farrugia, 2013). As mentioned in the literature, the low use of formal mathematical language by teachers can create problems such as hindering students from acquiring high-level knowledge (Simpson & Cole, 2015) and restricting their access to higher levels of language forms (Morgan, 2007). Therefore, teachers should use formal mathematical language more frequently in both describing mathematical situations and creating learning environments. The inclusion of formal mathematical language in learning environments will provide clarity for students in acquiring mathematical knowledge and provide opportunities to acquire higher-level knowledge.

The current research has shown that the mathematical language used by teachers for a given mathematical situation is influenced by the mathematical world in which it is presented. While mathematical language in "conceptual-embodied world" context is generally used by all teachers for mathematical situations, it is not used by all teachers in a "axiomatic-formal world" context. In other words, the use of mathematical language in "conceptual-embodied world" context to express formal situations has not been fully accomplished by teachers. Tall (2007) considers the "axiomatic-formal world" to be at the highest level of mathematical thinking processes. Therefore, reducing a given mathematical situation within a high-level thinking process to a simpler level of mathematical thinking process may have been challenging for teachers. Additionally, the research has shown that symbolic mathematical language is mostly used in both "conceptual-embodied world" and "proceptual-symbolic world" contexts of mathematical situations. Baiduri and Utomo (2020) have noted that teachers experience inconsistencies when using mathematical symbols, and Leiss et al. (2019) have stated that they struggle to express mathematical situations with symbols. Therefore, it can be said that teachers need a more linguistically rich representation of a mathematical situation when expressing it symbolically.

The results obtained within the scope of the study indicate that, from a general perspective, mathematics teachers have not been able to fully reflect mathematical language to different worlds when defining, expressing, and working on mathematical situations. While teachers have a good knowledge of mathematical concepts, they experience uncertainty in translating mathematical situations into appropriate mathematical language (Al-Sehli & Maroof, 2020). Studies in the literature also indicate that mathematics teachers have not been successful enough in explaining and expressing symbols and graphs accurately in connection with mathematical language (Burke, 2017; Ilany & Hassidov, 2018). In addition, there are studies in the literature that indicate that students have difficulties in using mathematical language (words, symbols, and signs) (Açıl & Zeybek, 2017) and that understanding mathematical language comprehensively continues to be a problem (Baiduri & Utomo, 2020). Considering these results, it is possible to mention the necessity of teachers using mathematical language in the learning environment to address different mathematical worlds.

Finally, within the scope of the current research, it has been concluded that the use of mathematical language by mathematics teachers in the “Understanding/Developing” dimension contains richer mathematical language contexts compared to the “Seeing/Perceiving” dimension. This indicates that mathematical language is more activated towards different contexts within the “Understanding/Developing” dimension. Teachers were asked to pose problems in the “Understanding/Developing” dimension. Problem posing requires a skill that involves higher-level thinking. During the performance of higher-level skills, mathematical language was exhibited by teachers in a more qualified manner. Based on this, activities that allow the development of higher-level skills in mathematics classes and that serve to understand and improve mathematics will enable teachers to use mathematical language in a richer way. It can be said that the instructional practices in which the teacher uses mathematical language richly, students will both increase their knowledge of mathematical language and improve their success in the class. Because instructional practices on mathematical language are an influential factor on student achievement (Firmender et al., 2014).

## References

Açıl, E. & Zeybek, Z. (2017). Öğrencilerin matematiksel dili kullanma ve anlama becerisi ile öğretmenlerinin öğrencilerin matematiksel dili nasıl kullandıklarını fark edebilme yeteneği. [Students’ ability to use and understand mathematical language with their teacher’s ability to notice the ways of how students use mathematical language?]. *Pamukkale University Journal of Education*, *42*(42), 87-107.

Akarsu Yakar, E. (2019). *An investigation of mathematical thinking processes and mathematical language skills of secondary school students through the theoretical framework of ‘Three Worlds of Mathematics’*. [Doctoral Dissertation]. Dokuz Eylül University, İzmir.

Akarsu, E. (2013). *The investigation of using mathematical language of 7th graders on algebra*. [Master Dissertation]. Dokuz Eylül University, İzmir.

Al-Sehli, L. J. & Maroof, N. (2020). Language barriers in teaching and understanding mathematics: Teachers' and students' perceptions. *Journal of Applied Linguistics and Language Research*, *7*(1), 125–134

Australian Curriculum & Assessment Authority [ACARA]. (2022, November 20). *Australian Curriculum: Mathematics.* https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/rationale/

Aydın, S. & Yeşilyurt, M. (2007). Matematik öğretiminde kullanılan dile ilişkin öğrenci görüşleri. [Student's view about the language used in mathematics teaching]. *Electronic Journal of Social Sciences, 22,* 90-100.

Aydoğan Belen, N. (2018). *The investigation of mathematical language of used by fourth grade students*. [Master Dissertation]. Ordu University, Ordu.

Baiduri, B. & Utomo, D. P. (2020). Teachers' competence in the acquisition and use of the mathematical language in Indonesia. *Humanities & Social Sciences Reviews, 8* (4), 1471-1481.

Baki, A. & Çelik, S. (2018). Veri işleme öğrenme alanına yönelik sınıf içindeki söylemlerin matematiksel dil bağlamında incelenmesi. [Investigation of Classroom Discourse about Statistics in the Context of Mathematical Language]. *Turkish Journal of Computer and Mathematics Education (TURCOMAT), 9*(2), 283-311.

Baltacı, A. (2017). Nitel veri analizinde Miles-Huberman modeli. [Miles-Huberman Model in qualitative data analysis]. *Ahi Evran University Institute of Social Sciences Journal*, *3*(1), 1-15.

Baltacı, A. (2019). Nitel araştırma süreci: Nitel bir araştırma nasıl yapılır?. [The Qualitative Research Process: How to Conduct a Qualitative Research?]. *Ahi Evran University Institute of Social Sciences Journal*, *5*(2), 368-388.

Bogdan, R. & Biklen, S. K. (1997). *Qualitative research for education*. Allyn & Bacon Boston, MA.

Burke, R. (2017). Language and culture in the mathematics classroom: Scaffolding learner engagement. Numeracy in Authentic Contexts: Making Meaning Across the Curriculum, s. 91–109.

Chard, D. (2003). Vocabulary strategies for the mathematics classroom. Date of access: 20.04.2023, https://pdf4pro.com/fullscreen/vocabulary-strategies-for-the-mathematics-ce4f9.html

Creswell, J.W. (2012). *Educational research: planning, conducting, and evaluating quantitative research*. Boston: Pearson Education Inc.

Çakmak, Z., Baş, F. & Bekdemir, M. (2014). İlköğretim matematik öğretmen adaylarının örüntüler konusundaki matematiksel dil becerileri. [Elementary mathematics education students' mathematical language skills on the issue of patterns]. *Erzincan University Journal of Education Faculty*, *16*(1), 204-223.

Çalıkoğlu Bali, G. (2002). Matematik öğretiminde dil ölçeği. [Language in mathematics teaching scale]. *H. U. Journal of Education, 23,* 57-61.

Denzin, N. K. & Lincoln, Y. S. (2008). *Introduction: The discipline and practice of qualitative research.*

Emre, E., Sağ, Y. G., Gülkılık, H. & Argün, Z. (2010). Matematik öğretmen adaylarının matematiksel dil kullanımları. [Mathematics teacher candidates' use of mathematical language]. *UFBMEK-2010.* Date of access: 12.04.2023, https://www.researchgate.net/publication/322056933_Matematik_ogretmen_adaylarinin_matematiksel_dil_kullanimlari

Farrugia, M. T. (2013). Moving from informal to formal mathematical language in Maltese classrooms. *International Journal of Bilingual Education and Bilingualism*, *16*(5), 570-588.

Firmender, J. M., Gavin, M. K., & McCoach, D. B. (2014). Examining the relationship between teachers' instructional practices and students' mathematics achievement. *Journal of Advanced Academics*, *25*(3), 214-236.

Gawned, S. (1990). An emerging model of the language of mathematics. *Language in mathematics*, 27-42.

Gültekin, S. H. & Es, H. (2018). İlköğretim matematik öğretmen adaylarının geometri alan dilini kullanma becerilerinin incelenmesi. [An analysis of the prospective primary school mathematics teachers' use of geometry field language skills among]. *Gazi University Gazi Faculty of Education Journal*, *38*(2), 635-662.

Gürefe, N. (2018). Mathematical language skills of mathematics prospective teachers. *Universal Journal of Educational Research*, *6*(4), 661-671.

Güzel, S. & Yılmaz, S. (2020). 8. Sınıf öğrencilerinin üslü ifadeler konusundaki matematiksel dil kullanım düzeyleri ve dile ilişkin görüşleri. [Mathematical language usage levels of eighth grade students regarding exponential expression and their views on language]. *International Journal of Active Learning*, *5*(1), 33-56.

Hancock, D.R. & Algozzine, B. (2006). *Doing case study research: A practical guide for beginners researchers.* New York: Teachers College.

Harvey, R. (1982). I can keep going up if I want to': one way of looking at learning in mathematics. In *Mathematics*, ed. R. Harvey, D. Kerslake, H. Shuard, and M. Torbe, 2240. London: Ward Lock Educational.

Hornburg, C. B., Schmitt, S. A. & Purpura, D. J. (2018). Relations between preschoolers' mathematical language understanding and specific numeracy skills. *Journal of experimental child psychology*, *176*, 84-100.

Ilany, B. S. & Hassidov, D. (2018). Perspective of "Quantity" between natural language and mathematical symbols (<,>,=): The comprehension of pre-service and preschool teachers. *Creative Education*, *9*(10), 1525.

Jupri, A., & Drijvers, P. (2016). Student difficulties in mathematizing word problems in algebra. *Eurasia Journal of Mathematics, Science and Technology Education*, *12*(9), 2481-2502.

Kıymaz, Y., Kartal, B., & Morkoyunlu, Z. (2020). İlköğretim matematik öğretmen adaylarının yazılı matematiksel iletişim becerilerinin incelenmesi. [Examining pre-service mathematics teachers' written mathematical communication skills]. *Journal of Uludag University Faculty of Education*, *33*(1), 205-228.

Korhonen, J., Linnanmäki, K. & Aunio, P. (2011). Language and mathematical performance: a comparison of lower secondary school students with different level of mathematical skills, *Scandinavian Journal of Educational Research,* 1-12.

Kuntze, S., Prinz, E., Friesen, M., Batzel-Kremer, A., Bohl, T., & Kleinknecht, M. (2018). Using multiple representations as part of the mathematical language in classrooms: investigating teachers' support in a video analysis. In *Proceedings of the IV ERME Topic Conference'Classroom-based research on mathematics and language (p. 96-102).*

Leiss, D., Plath, J., & Schwippert, K. (2019). Language and mathematics-key factors influencing the comprehension process in reality-based tasks. *Mathematical Thinking and Learning*, *21*(2), 131-153.

Maxwell, J. A. (2008). Designing a qualitative study. *The SAGE handbook of applied social research methods*, *2*, 214-253.

McClelland, M. M., Cameron, C. E., Connor, C. M., Farris, C. L., Jewkes, A. M., & Morrison, F. J. (2007). Links between behavioral regulation and preschoolers' literacy, vocabulary, and math skills. *Developmental psychology*, *43*(4), 947.

Mercer, N. & Sams, C. (2006). Teaching children how to use language to solve maths problems. *Language and Education*, *20*(6), 507-528.

Merriam, S. B. (1998). *Qualitative Research and Case Study Applications in Education. Revised and Expanded from" Case Study Research in Education."*. Jossey-Bass Publishers.

Ministry of National Education [MoNE]. (2009). *Elementary math lesson 6-8. classes curriculum and guide*. Ministry of National Education.

Ministry of National Education [MoNE]. (2018). *School mathematics curriculum (Primary and Secondary School Grades 1, 2, 3, 4, 5, 6, 7 and 8)*. Ministry of National Education.

Morgan, C. (2007). Who is not multilingual now?. *Educational Studies in Mathematics*, *64*, 239-242.

Morgan, D. L. (1996). *Focus groups as qualitative research* (C. 16). New York: Sage publications.

Mulwa, E. C. (2015). Difficulties encountered by students in the learning and usage of mathematical terminology: A critical literature review. *Journal of Education and Practice*, *6*(13), 27-37.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. National Council of Mathematics Teachers.

Oxenford O'Brian, J. R., Nocon, H., & Iceman Sands, D. (2010). The use of dialogue and tools to develop students' mathematical language and meta-cognition. *Teacher Development*, *14*(4), 447-466.

Örnek, T., & Soylu, Y. (2021). A model design to be used in teaching problem posing to develop problem-posing skills. *Thinking Skills and Creativity*, *41*, 100905.

Patton, M. Q. (2018). Bütün, M. & Demir, S. B. (Eds.). (2018). *Nitel araştırma ve değerlendirme yöntemleri*. [*Qualitative research and evaluation methods*]. Pegem Akademi.

Purpura, D. J. & Reid, E. E. (2016). Mathematics and language: Individual and group differences in mathematical language skills in young children. *Early Childhood Research Quarterly*, *36*, 259-268.

Purpura, D. J., & Logan, J. A. R. (2015). The non-linear relations between the approximate number system and mathematical language to symbolic mathematics. *Developmental Psychology,* 51, 1717–1724.

Rubenstein, R. N., & Thompson, D. R. (2002). Understanding and supporting children's mathematical vocabulary development. *Teaching Children Mathematics*, *9*, 107–112.

Rudd, L. C., Lambert, M. C., Satterwhite, M. & Zaier, A. (2008). Mathematical language in early childhood settings: What really counts?, *Early Childhood Education*, *36*, 75-80.

Rudnitsky, A., Etheredge, S., Freeman, S. J. M. & Gilbert, T. (1995). Learning to solve addition and substraction word problems through a structure-plus-writing approach. *Journal for Research in Mathematics Education*, *26*(5), 467-486.

Schmitt, S. A., Purpura, D. J., & Elicker, J. G. (2019). Predictive links among vocabulary, mathematical language, and executive functioning in preschoolers. *Journal of experimental child psychology*, *180*, 55-68.

Shockey, T., & Pindiprolu, S. (2015). Uniquely precise: Importance of conceptual knowledge and mathematical language. *Journal on School Educational Technology*, *11*(1), 28-33.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher*, *15*(2), 4-14.

Simpson, A., & Cole, M. W. (2015). More than words: A literature review of language of mathematics research. *Educational Review*, *67*(3), 369-384.

Singleton, R.A. & Straits, B.C. (2005). *Approaches to social research* (4th ed.). New York: Oxford University Press.

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, *20*(2), 5-24.

Tall, D. O. (2007). Developing a theory of mathematical growth. *ZDM*, *39*(1), 145-154.

Teddlie, C., & Yu, F. (2007). Mixed methods sampling: A typology with examples. *Journal of mixed methods research*, *1*(1), 77-100.

Uğurel, I. & Moralı, H. S. (2010). Bir ortaöğretim matematik dersindeki ispat yapma etkinliğine yönelik sınıf içi tartışma sürecine öğrenci söylemleri çerçevesinde yakından bakış. [A close view on the discussion in relation to a activity about proving a theorem in a high school mathematics lesson via students' discourse]. *Journal of Buca Faculty of Education*, (28), 134-154.

Yakar, E. A. & Yılmaz, S. (2017). 7. sınıf öğrencilerinin cebire yönelik gerçek yaşam durumlarını matematiksel ifadelere dönüştürme sürecindeki matematiksel dil becerileri. [Mathematical language skills of 7th grade students in the process of transforming the real life situation into a mathematical expression in algebra]. *Inonu University Journal of the Faculty of Education*, *18*(1), 292-310.

Yardımcı, H. (2019). *Mathematics teacher candidates' ability to convert verbally expressed sets into mathematical language*. [Master Dissertation]. Mersin University, Mersin.

Yeşildere, S. (2007). İlköğretim matematik öğretmen adaylarının matematiksel alan dilini kullanma yeterlikleri. [The competencies of primary school mathematics teacher candidates in using the mathematical field language]. *Boğaziçi University Journal of Education*, *24*(2), 61-70.

Yıldırım, A. & Şimşek, H. (2021). *Sosyal bilimlerde nitel araştırma yöntemleri. [Qualitative research methods in the social sciences].* Ankara: Seçkin Yayıncılık.

Yin, R. K. (2014). *Case study research: design and methods.* Sage Pbc.

Yüzerler, S. (2013). *6th and 7th grade students' skills of using mathematical language.* [Doctoral Dissertation]. Necmettin Erbakan University, Konya.