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New Generalized Fixed Point Results on S_b-Metric Spaces

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Abstract

Recently, S_b-metric spaces have been introduced as the generalizations of metric and S-metric spaces. In this paper, we generalize the classical Banach contraction principle using the theory of a complete S_b-metric space. Also, we give an application to linear equation systems using the S_b-metric generated by a metric.

Keywords: *S_b-metric space, fixed point, the Banach contraction principle, linear equation.*
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1. Introduction and mathematical preliminaries

Metric spaces and fixed point theorems are very important in many areas of mathematics. Some generalizations of metric spaces and fixed points of various contractive mappings have been studied extensively. Bakhtin introduced b-metric spaces as a generalization of metric spaces [5]. Mustafa and Sims defined the concept of a generalized metric space which is called a G-metric space [17]. Sedghi, Shobe and Aliouche gave the notion of an S-metric space and proved some fixed-point theorems for a self-mapping on a complete S-metric space [23]. Aghajani, Abbas and Roshan presented a new type of metric which is called G_b-metric and studied some properties of this metric [1]. Since then, many authors obtained several fixed-point results in the various generalized metric spaces (see [2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 15, 18, 19, 20, 24, 25] for more details). Also, some applications of fixed point theory were studied on various metric spaces. Several applications of the Banach contraction principle were given in many areas such as integral equations, linear equations, differential equations etc. For example, the present authors investigated some applications on S-metric spaces (see [21] and [22]).
Recently, the concept of an S_b-metric space, as a generalization of metric spaces and S-metric spaces, has been introduced in [26] and a common fixed point theorem for four mappings has been studied on a complete S_b-metric space. The notion of an S_b-metric was generalized to the notion of an A_b-metric in [29]. When $n = 3$, the notion of "an S_b-metric" coincides with the notion of "an A_b-metric". Some fixed point theorems were given under different contraction and expansion type conditions (see [29] for more details). After then, some fixed-point results have been studied with various approaches (see [13, 14, 16, 27, 30] for some examples).
In this paper, we consider a complete S_b-metric space and prove two generalizations of the classical Banach fixed point result. In Section 2, we recall some known definitions. In Section 3, we deal with the notion of an S_b-metric and investigate some properties of S_b-metric spaces. We study some relationships between an S_b-metric and some other metrics. In Section 4, we prove the Banach contraction principle on a complete S_b-metric space and give a new fixed point theorem as a generalization of the Banach contraction principle with a counterexample. In Section 5, we present an application to linear equations on an S_b-metric space (X,S_1).
Now we recall the following definitions.

Definition 1.1. *[5] Let X be a nonempty set, $b \geq 1$ a given real number and $d : X \times X \to [0,\infty)$ a function satisfying the following conditions for all $x,y,z \in X$:*
$(b1)$ $d(x,y) = 0$ *if and only if* $x = y$.
$(b2)$ $d(x,y) = d(y,x)$.
$(b3)$ $d(x,z) \leq b[d(x,y) + d(y,z)]$.
Then the function d is called a b-metric on X and the pair (X,d) is called a b-metric space.

Definition 1.2. *[17] Let X be a nonempty set and $G : X \times X \times X \to [0,\infty)$ a function satisfying the following conditions*:
$(G1)$ $G(x,y,z) = 0$ *if* $x = y = z$.
$(G2)$ $0 < G(x,x,y)$ *for all* $x,y \in X$ *with* $x \neq y$.
$(G3)$ $G(x,x,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$ *with* $y \neq z$.

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$(G4)$ $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$.
$(G5)$ $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$.
Then the function G *is called a generalized metric or a G-metric on* X *and the pair* (X,G) *is called a G-metric space.*

Definition 1.3. *[1] Let* X *be a nonempty set,* $b \geq 1$ *a given real number and* $G_b : X \times X \times X \to [0,\infty)$ *a function satisfying the following conditions*:
(G_b1) $G_b(x,y,z) = 0$ *if* $x = y = z$.
(G_b2) $0 < G_b(x,x,y)$ *for all* $x,y \in X$ *with* $x \neq y$.
(G_b3) $G_b(x,x,y) \leq G_b(x,y,z)$ *for all* $x,y,z \in X$ *with* $y \neq z$.
(G_b4) $G_b(x,y,z) = G_b(x,z,y) = G_b(y,z,x) = \cdots$.
(G_b5) $G_b(x,y,z) \leq b[G_b(x,a,a) + G_b(a,y,z)]$ *for all* $x,y,z,a \in X$.
Then the function G_b *is called a generalized b-metric or a* G_b*-metric on* X *and the pair* (X,G_b) *is called a* G_b*-metric space.*

Definition 1.4. *[23] Let* X *be a nonempty set and* $S : X \times X \times X \to [0,\infty)$ *a function satisfying the following conditions for all* $x,y,z,a \in X$:
$(S1)$ $S(x,y,z) = 0$ *if and only if* $x = y = z$.
$(S2)$ $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.
Then the function S *is called an S-metric on* X *and the pair* (X,S) *is called an S-metric space.*

We use the following lemma in the next sections.

Lemma 1.5. *[23] Let* (X,S) *be an S-metric space. Then we have*

$$S(x,x,y) = S(y,y,x).$$

2. S_b-Metric spaces

In this section, we recall the notion of an S_b-metric space and study some properties of this space.

Definition 2.1. *[26] Let* X *be a nonempty set and* $b \geq 1$ *a given real number. A function* $S_b : X \times X \times X \to [0,\infty)$ *is said to be* S_b*-metric if and only if for all* $x,y,z,a \in X$ *the following conditions are satisfied*:
(S_b1) $S_b(x,y,z) = 0$ *if and only if* $x = y = z$,
(S_b2) $S_b(x,y,z) \leq b[S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a)]$.
The pair (X,S_b) *is called an* S_b*-metric space.*
We note that S_b*-metric spaces are the generalizations of S-metric spaces since every S-metric is an* S_b*-metric with* $b = 1$*. But the converse statement is not always true (see [26] for more details). In the following, we give another example of an* S_b*-metric which is not an S-metric on* X.

Example 2.2. *Let* $X = \mathbb{R}$ *and the function* S_b *be defined as*

$$S_b(x,y,z) = \frac{1}{16}(|x-y| + |y-z| + |x-z|)^2.$$

Then the function S_b *is an* S_b*-metric with* $b = 4$*, but it is not an S-metric. Indeed, for* $x = 4$*,* $y = 6$*,* $z = 8$ *and* $a = 5$*, we get*

$$S_b(4,6,8) = 4,\ S_b(4,4,5) = \frac{1}{4},\ S_b(6,6,5) = \frac{1}{4},\ S_b(8,8,5) = \frac{9}{4}.$$

Hence we have

$$S_b(4,6,8) = 4 \leq S_b(4,4,5) + S_b(6,6,5) + S_b(8,8,5) = \frac{11}{4},$$

which is a contradiction with $(S2)$.

Definition 2.3. *Let* (X,S_b) *be an* S_b*-metric space and* $b > 1$*. An* S_b*-metric* S_b *is called symmetric if*

$$S_b(x,x,y) = S_b(y,y,x), \tag{2.1}$$

for all $x,y \in X$.

In [28], it was given a definition of an S_b-metric with the symmetry condition "$S_b(x,x,y) = S_b(y,y,x)$" (see Definition 1.3 on page 132). However, in the definition of an S_b-metric, the symmetry condition (2.1) is not necessary. In fact, for $b = 1$ the S_b-metric induced to an S-metric. It is known that the symmetry condition (2.1) is automatically satisfied by an S-metric (see Lemma 1.5). So Definition 2.1 of an S_b-metric is more general than given in [28].
We give the following examples of a symmetric S_b-metric and a non-symmetric S_b-metric, respectively.

Example 2.4. *Let* (X,d) *be a metric space and the function* $S_b : X \times X \times X \to [0,\infty)$ *defined as*

$$S_b(x,y,z) = [d(x,y) + d(y,z) + d(x,z)]^p,$$

for all $x,y,z \in X$ *and* $p > 1$*. Then it can be easily seen that* S_b *is an* S_b*-metric on* X*. Also the function* S_b *satisfies the symmetry condition (2.1).*

Example 2.5. *Let $X=\mathbb{R}$ and the function $S_b : X\times X\times X \to [0,\infty)$ be defined as*

$S_b(0,0,1)=2,$
$S_b(1,1,0)=4,$
$S_b(x,y,z)=0$ *if* $x=y=z,$
$S_b(x,y,z)=1$ *otherwise,*

for all $x,y,z\in\mathbb{R}$. Then the function S_b is an S_b-metric with $b\geq 2$ which is not symmetric.

We define some topological concepts in the following:

Definition 2.6. *Let (X,S_b) be an S_b-metric space, $x\in X$ and $A,B\subset X$.*

1. *We define the distance between the sets A and B by*

   $S_b(A,A,B)=\inf\{S_b(x,x,y): x\in A, y\in B\}.$
2. *We define the distance of the point x to the set A by*

   $S_b(x,x,A)=\inf\{S_b(x,x,y): y\in A\}.$
3. *We define the diameter of A by*

   $\delta(A)=\sup\{S_b(x,x,y): x,y\in A\}.$

Now we recall the definition of an open ball and a closed ball on S_b-metric spaces, respectively.

Definition 2.7. *[26] Let (X,S_b) be an S_b-metric space. The open ball $B_S^b(x,r)$ and the closed ball $B_S^b[x,r]$ with a center x and a radius r are defined by*

$B_S^b(x,r)=\{y\in X : S_b(y,y,x)<r\}$

and

$B_S^b[x,r]=\{y\in X : S_b(y,y,x)\leq r\},$

for $r>0$, $x\in X$, respectively.

Example 2.8. *Let us consider the S_b-metric space defined in Example 2.2 as follows:*

$$S_b(x,y,z)=\frac{1}{16}(|x-y|+|y-z|+|x-z|)^2,$$

for all $x,y,z\in\mathbb{R}$. Then we get

$B_S^b(0,2)=\{y\in\mathbb{R} : S_b(y,y,0)<2\}=(-2\sqrt{2},2\sqrt{2})$

and

$B_S^b[0,2]=\{y\in\mathbb{R} : S_b(y,y,0)\leq 2\}=[-2\sqrt{2},2\sqrt{2}].$

Definition 2.9. *Let (X,S_b) be an S_b-metric space and $X'\subset X$.*

1. *If there exists an $r>0$ such that $B_S^b(x,r)\subset X'$ for every $x\in X'$ then X' is called an open subset of X.*
2. *Let τ be the set of all $X'\subset X$ with $x\in X'$ such that there exists an $r>0$ satisfying $B_S^b(x,r)\subset X'$. Then τ is called the topology induced by the S_b-metric.*
3. *X' is called S_b-bounded if there exists an $r>0$ such that $S_b(x,x,y)<r$ for all $x,y\in X'$. If X' is S_b-bounded then we will write $\delta(X')<\infty$.*

Definition 2.10. *[26] Let (X,S_b) be an S_b-metric space.*

1. *A sequence $\{x_n\}$ in X converges to x if and only if $S_b(x_n,x_n,x)\to 0$ as $n\to\infty$, that is, for each $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, $S_b(x_n,x_n,x)<\varepsilon$. It is denoted by*

   $\lim_{n\to\infty} x_n = x.$
2. *A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that $S_b(x_n,x_n,x_m)<\varepsilon$ for each $n,m\geq n_0$.*
3. *The S_b-metric space (X,S_b) is said to be complete if every Cauchy sequence is convergent.*

Now we investigate some relationships between S_b-metric and some other metrics. The relationship between a metric and an S-metric are given in [11] as follows:

Lemma 2.11. *[11] Let (X,d) be a metric space. Then the following properties are satisfied:*

1. *$S_d(x,y,z)=d(x,z)+d(y,z)$ for all $x,y,z\in X$ is an S-metric on X.*
2. *$x_n\to x$ in (X,d) if and only if $x_n\to x$ in (X,S_d).*
3. *$\{x_n\}$ is Cauchy in (X,d) if and only if $\{x_n\}$ is Cauchy in (X,S_d).*
4. *(X,d) is complete if and only if (X,S_d) is complete.*

Since every S-metric is an S_b-metric, using Lemma 2.11, an S_b-metric generated by a metric d is defined as follows:

$S_b^d(x,y,z) = b[d(x,z) + d(y,z)]$,

for all $x, y, z \in X$ with $b \geq 1$. But there exists an S_b-metric which is not generated by any metric as seen in the following example.

Example 2.12. *Let $X = \mathbb{R}$. We consider the function $S : X \times X \times X \to [0,\infty)$ given in [19] as follows:*

$S(x,y,z) = |x - z| + |x + z - 2y|$,

for all $x, y, z \in \mathbb{R}$. Then (X,S) is an S-metric space. Hence (X,S) is an S_b-metric space with $b = 1$. This metric is not generated by any metric d.

In the following lemmas, we show that the relationships between a b-metric and an S_b-metric.

Lemma 2.13. *Let (X,S_b) be an S_b-metric space, S_b a symmetric S_b-metric with $b \geq 1$ and the function $d : X \times X \to [0,\infty)$ defined by*

$d(x,y) = S_b(x,x,y)$,

for all $x, y \in X$. Then d is a b-metric on X.

Proof. It can be easily seen that the conditions $(b1)$ and $(b2)$ are satisfied. Now we show that the condition $(b3)$ is satisfied. Using the inequality (S_b2), we have

$$\begin{aligned} d(x,y) &= S_b(x,x,y) \leq b[2S_b(x,x,z) + S_b(y,y,z)] \\ &= 2bS_b(x,x,z) + bS_b(y,y,z) \end{aligned}$$

and

$$\begin{aligned} d(x,y) &= S_b(y,y,x) \leq b[2S_b(y,y,z) + S_b(x,x,z)] \\ &= 2bS_b(y,y,z) + bS_b(x,x,z). \end{aligned}$$

Hence we obtain

$d(x,y) \leq \dfrac{3b}{2}[d(x,z) + d(y,z)]$,

for all $x, y \in X$. Then d is a b-metric on X with $\dfrac{3b}{2}$. □

Lemma 2.14. *Let (X,d) be a b-metric space with $b \geq 1$ and the function $S_b : X \times X \times X \to [0,\infty)$ be defined by*

$S_b(x,y,z) = d(x,z) + d(y,z)$,

for all $x, y, z \in X$. Then S_b is an S_b-metric on X.

Proof. It can be easily verified that the condition (S_b1) is satisfied. We prove that the condition (S_b2) is satisfied. Using the inequality $(b3)$ we get

$$\begin{aligned} S_b(x,y,z) &= d(x,z) + d(y,z) \\ &\leq b[d(x,a) + d(a,z)] + b[d(y,a) + d(a,z)] \\ &= bd(x,a) + 2bd(a,z) + bd(y,a) \\ &\leq 2bd(x,a) + 2bd(y,a) + 2bd(a,z) \\ &= b[S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a)], \end{aligned}$$

for all $x, y, z \in X$. Then S_b is an S_b-metric on X with b. □

Now we give the following example to show that there exists an S_b-metric which is not generated by any b-metric.

Example 2.15. *Let $X = \mathbb{R}$ and define the function $S_b : X \times X \times X \to [0,\infty)$*

$S_b(x,y,z) = b(|x - z| + |x + z - 2y|)$,

for all $x, y, z \in \mathbb{R}$, where $b \geq 1$. Then $(\mathbb{R},S_b)$ is an S_b-metric space. Now we show that there does not exist any b-metric d which generates this S_b-metric. Conversely, assume that there exists a b-metric d such that

$S_b(x,y,z) = d(x,z) + d(y,z)$,

for all $x, y, z \in \mathbb{R}$. Then we get

$S_b(x,x,z) = 2d(x,z) = 2b|x - z|$ *and* $d(x,z) = b|x - z|$

and

$S_b(y,y,z) = 2d(y,z) = 2b|y - z|$ *and* $d(y,z) = b|y - z|$,

for all $x, y, z \in \mathbb{R}$. Therefore we obtain

$b(|x - z| + |x + z - 2y|) = b|x - z| + b|y - z|$,

which is a contradiction. Consequently, the S_b-metric can not be generated by any b-metric.

Remark 2.16. *Notice that the class of all S-metrics and the class of all G-metrics are distinct [6]. Since every S-metric is an S_b-metric and every G-metric is a G_b-metric then the class of all S_b-metrics and the class of all G_b-metrics are distinct.*

3. Some fixed point results

In this section, we prove the Banach contraction principle on complete S_b-metric spaces. Then we give a generalization of this principle. We use the following lemma.

Lemma 3.1. *[26] Let (X,S_b) be an S_b-metric space with $b \geq 1$, then we have*

$S_b(x,x,y) \leq bS_b(y,y,x)$ *and* $S_b(y,y,x) \leq bS_b(x,x,y)$.

Theorem 3.2. *Let (X,S_b) be a complete S_b-metric space with $b \geq 1$ and $T: X \to X$ a self-mapping satisfying*

$$S_b(Tx,Tx,Ty) \leq hS_b(x,x,y), \tag{3.1}$$

for all $x,y,z \in X$, where $0 \leq h < \frac{1}{b^2}$. Then T has a unique fixed point x in X.

Proof. Let T satisfies the inequality (3.1) and $x_0 \in X$. Then we define the sequence $\{x_n\}$ by $x_n = T^n x_0$. Using the inequality (3.1) and mathematical induction, we obtain

$$S_b(x_n,x_n,x_{n+1}) \leq h^n S_b(x_0,x_0,x_1). \tag{3.2}$$

Since the conditions (S_b2) and (3.2) are satisfied for all $n,m \in \mathbb{N}$ with $m > n$, using Lemma 3.1 we get

$$\begin{array}{rcl} S_b(x_n,x_n,x_m) & \leq & b[2S_b(x_n,x_n,x_{n+1}) + S_b(x_m,x_m,x_{n+1})] \\ & \leq & b\,[2S_b(x_n,x_n,x_{n+1}) + bS_b(x_{n+1},x_{n+1},x_m)] \\ & \leq & 2bS_b(x_n,x_n,x_{n+1}) + b^3\,[2S_b(x_{n+1},x_{n+1},x_{n+2}) + S_b(x_m,x_m,x_{n+2})] \\ & \leq & 2bS_b(x_n,x_n,x_{n+1}) + b^3\,[2S_b(x_{n+1},x_{n+1},x_{n+2}) + bS_b(x_{n+2},x_{n+2},x_m)] \\ & \leq & 2bS_b(x_n,x_n,x_{n+1}) + 2b^3 S_b(x_{n+1},x_{n+1},x_{n+2}) + b^4 S_b(x_{n+2},x_{n+2},x_m) \\ & & \cdots \\ & \leq & 2bS_b(x_n,x_n,x_{n+1}) + 2b^3 S_b(x_{n+1},x_{n+1},x_{n+2}) + \cdots + 2b^{2m-2n-1} S_b(x_{m-1},x_{m-1},x_m) \\ & \leq & \left(2bh^n + 2b^3h^{n+1} + \ldots + 2b^{2m-2n-1}h^{m-1}\right) S_b(x_0,x_0,x_1) \\ & \leq & 2bh^n \left(1 + b^2h + b^4h^2 + \ldots + b^{2m-2n-2}h^{m-n-1}\right) S_b(x_0,x_0,x_1) \\ & = & 2bh^n \dfrac{1 - b^{2m-2n}h^{m-n}}{1-b^2h} S_b(x_0,x_0,x_1) \\ & \leq & \dfrac{2bh^n}{1-b^2h} S_b(x_0,x_0,x_1). \end{array}$$

Since $h \in \left[0, \frac{1}{b^2}\right)$, where $b \geq 1$, taking limit for $n \to \infty$ then we obtain $S_b(x_n,x_n,x_m) \to 0$ and so $\{x_n\}$ is a Cauchy sequence. Since X is complete S_b-metric space there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$.
Assume that $Tx \neq x$. Using the inequality (3.1) we have

$$S_b(Tx,Tx,x_{n+1}) \leq hS_b(x,x,x_n).$$

If we take limit for $n \to \infty$, we get a contradiction as follows:

$$S_b(Tx,Tx,x) \leq hS_b(x,x,x).$$

Hence $Tx = x$. Now we show that the fixed point x is unique. Suppose that $Tx = x$, $Ty = y$ and $x \neq y$. Using the inequality (3.1), we have

$$S_b(Tx,Tx,Ty) = S_b(x,x,y) \leq hS_b(x,x,y).$$

We obtain $x = y$ since $h \in \left[0, \frac{1}{b^2}\right)$. Consequently, x is a unique fixed point of the self-mapping T. □

Remark 3.3. *If we take $b = 1$ in Theorem 3.2 then we obtain Theorem 1 in [20].*

Corollary 3.4. *Let (X,S_b) be a complete S_b-metric space with $b \geq 1$, S_b symmetric and $T: X \to X$ a self-mapping satisfying the inequality (3.1) for all $x,y,z \in X$, where $0 \leq h < \frac{1}{b}$. Then T has a fixed point x in X.*

Proof. In the proof of Theorem 3.2, if we use the symmetry condition (2.1) instead of Lemma 3.1, we obtain

$$\begin{aligned} S_b(x_n,x_n,x_m) &\leq b[2S_b(x_n,x_n,x_{n+1})+S_b(x_m,x_m,x_{n+1})] \\ &= b\left[2S_b(x_n,x_n,x_{n+1})+S_b(x_{n+1},x_{n+1},x_m)\right] \\ &\leq 2bS_b(x_n,x_n,x_{n+1})+b^2\left[2S_b(x_{n+1},x_{n+1},x_{n+2})+S_b(x_m,x_m,x_{n+2})\right] \\ &= 2bS_b(x_n,x_n,x_{n+1})+2b^2S_b(x_{n+1},x_{n+1},x_{n+2})+b^2S_b(x_{n+2},x_{n+2},x_m) \\ &\ldots \\ &= 2bS_b(x_n,x_n,x_{n+1})+2b^2S_b(x_{n+1},x_{n+1},x_{n+2})+\ldots+2b^{m-n}S_b(x_{m-1},x_{m-1},x_m) \\ &\leq \left(2bh^n+2b^2h^{n+1}+\ldots+2b^{m-n}h^{m-1}\right)S_b(x_0,x_0,x_1) \\ &\leq 2bh^n\left(1+bh+b^2h^2+\ldots+b^{m-n-1}h^{m-n-1}\right)S_b(x_0,x_0,x_1) \\ &\leq 2bh^n\frac{1-b^{m-n}h^{m-n}}{1-bh}S_b(x_0,x_0,x_1) \\ &\leq \frac{2bh^n}{1-hb}S_b(x_0,x_0,x_1). \end{aligned}$$

Since $h\in\left[0,\frac{1}{b}\right)$ with $b\geq 1$, the rest of the proof is similar to that in the proof of Theorem 3.2. □

Example 3.5. *Let $X=\mathbb{R}$ and consider the S_b-metric defined in Example 2.2 as follows:*

$$S_b(x,y,z)=\frac{1}{16}(|x-y|+|y-z|+|x-z|)^2,$$

for all $x,y,z\in\mathbb{R}$ with $b=4$. If we define the self-mapping T of $\mathbb{R}$ as

$$Tx=\frac{x}{6},$$

for all $x\in\mathbb{R}$ then T satisfies the condition of the Banach contraction principle. Indeed, we get

$$S_b(Tx,Tx,Ty)=\frac{|x-y|^2}{144}\leq hS_b(x,x,y)=\frac{|x-y|^2}{72},$$

for all $x\in\mathbb{R}$ and $h=\frac{1}{18}$. Hence T has a unique fixed point $x=0$ in $\mathbb{R}$.

Now we give the following theorem as a generalization of the Banach contraction principle on complete S_b-metric spaces.

Theorem 3.6. *Let (X,S_b) be a complete S_b-metric space with $b\geq 1$ and T a self-mapping of X satisfying the following condition: There exist real numbers α_1, α_2 satisfying $\alpha_1+\left(2b^2+b\right)\alpha_2<1$ with α_1, $\alpha_2\geq 0$ such that*

$$S_b(Tx,Tx,Ty) \leq \alpha_1S_b(x,x,y)+\alpha_2\max\{S_b(Tx,Tx,x),S_b(Tx,Tx,y),S_b(Ty,Ty,y),S_b(Ty,Ty,x)\}, \tag{3.3}$$

for all $x,y\in X$. Then T has a unique fixed point x in X.

Proof. Let $x_0\in X$ and the sequence $\{x_n\}$ be defined as follows:

$$Tx_0=x_1,\,Tx_1=x_2,\,\cdots,\,Tx_n=x_{n+1},\,\cdots.$$

Assume that $x_n\neq x_{n+1}$ for all n. Using the condition (3.3), we get

$$\begin{aligned} S_b(x_n,x_n,x_{n+1}) &= S_b(Tx_{n-1},Tx_{n-1},Tx_n)\leq \alpha_1S_b(x_{n-1},x_{n-1},x_n) \\ &\quad +\alpha_2\max\{S_b(x_n,x_n,x_{n-1}),S_b(x_n,x_n,x_n), \\ &\quad S_b(x_{n+1},x_{n+1},x_n),S_b(x_{n+1},x_{n+1},x_{n-1})\} \\ &= \alpha_1S_b(x_{n-1},x_{n-1},x_n)+\alpha_2\max\{S_b(x_n,x_n,x_{n-1}), \\ &\quad S_b(x_{n+1},x_{n+1},x_n),S_b(x_{n+1},x_{n+1},x_{n-1})\}. \end{aligned} \tag{3.4}$$

By the condition (S_b2), we have

$$S_b(x_{n+1},x_{n+1},x_{n-1})\leq b[2S_b(x_{n+1},x_{n+1},x_n)+S_b(x_{n-1},x_{n-1},x_n)]. \tag{3.5}$$

Using the conditions (3.4), (3.5) and Lemma 3.1, we obtain

$$\begin{aligned} S_b(x_n,x_n,x_{n+1}) &\leq \alpha_1S_b(x_{n-1},x_{n-1},x_n)+\alpha_2\max\{S_b(x_n,x_n,x_{n-1}), \\ &\quad S_b(x_{n+1},x_{n+1},x_n),2bS_b(x_{n+1},x_{n+1},x_n)+bS_b(x_{n-1},x_{n-1},x_n) \\ &\leq \alpha_1S_b(x_{n-1},x_{n-1},x_n)+2b\alpha_2S_b(x_{n+1},x_{n+1},x_n) \\ &\quad +b\alpha_2S_b(x_{n-1},x_{n-1},x_n) \\ &\leq \alpha_1S_b(x_{n-1},x_{n-1},x_n)+2b^2\alpha_2S_b(x_n,x_n,x_{n+1}) \\ &\quad +b\alpha_2S_b(x_{n-1},x_{n-1},x_n) \end{aligned}$$

and so

$$(1-2b^2\alpha_2)S_b(x_n,x_n,x_{n+1}) \leq (\alpha_1+b\alpha_2)S_b(x_{n-1},x_{n-1},x_n),$$

which implies

$$S_b(x_n,x_n,x_{n+1}) \leq \frac{\alpha_1+b\alpha_2}{1-2b^2\alpha_2}S_b(x_{n-1},x_{n-1},x_n). \tag{3.6}$$

Let $d=\frac{\alpha_1+b\alpha_2}{1-2b^2\alpha_2}$. Then $d<1$ since $\alpha_1+\left(2b^2+b\right)\alpha_2<1$. Notice that $1-2b^2\alpha_2 \neq 0$ since $0 \leq \alpha_2 < \frac{1}{2b^2+b}$. Now repeating this process in the inequality (3.6), we get

$$S_b(x_n,x_n,x_{n+1}) \leq d^n S_b(x_0,x_0,x_1). \tag{3.7}$$

We show that the sequence $\{x_n\}$ is Cauchy. Indeed, for all $n,m \in \mathbb{N}$, $m>n$, using the conditions (3.7) and (S_b2), we obtain

$$S_b(x_n,x_n,x_m) \leq \frac{2bd^n}{1-b^2d}S_b(x_0,x_0,x_1).$$

We have $\lim\limits_{n,m\to\infty} S_b(x_n,x_n,x_m)=0$ by the above inequality and so $\{x_n\}$ is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that $\{x_n\}$ converges to x. Suppose that $Tx \neq x$. Then we have

$$\begin{aligned} S_b(x_n,x_n,Tx) &= S_b(Tx_{n-1},Tx_{n-1},Tx) \\ &\leq \alpha_1 S_b(x_{n-1},x_{n-1},x)+\alpha_2\max\{S_b(x_n,x_n,x_{n-1}), \\ &\quad S_b(x_n,x_n,x),S_b(Tx,Tx,x),S_b(Tx,Tx,x_{n-1})\} \end{aligned}$$

and so taking limit for $n \to \infty$ and using Lemma 3.1, we get

$$S_b(x,x,Tx) \leq \alpha_2 S_b(Tx,Tx,x) \leq \alpha_2 b S_b(x,x,Tx),$$

which implies $S_b(Tx,Tx,x)=0$ and $Tx=x$ since $0 \leq \alpha_2 < \frac{1}{2b^2+b}$.
Finally we show that the fixed point x is unique. To do this, we assume that $x \neq y$ such that $Tx=x$ and $Ty=y$. Using the inequality (3.3) and Lemma 3.1, we have

$$S_b(Tx,Tx,Ty) = S_b(x,x,y) \leq \alpha_1 S_b(x,x,y)+\alpha_2\max\{S_b(x,x,x),S_b(x,x,y),S_b(y,y,y),S_b(y,y,x)\},$$

which implies $x=y$ since $\alpha_1+b\alpha_2<1$. Then the proof is completed. □

Corollary 3.7. *Let (X,S_b) be a complete S_b-metric space with $b \geq 1$, S_b symmetric and T a self-mapping of X satisfying the following condition:*
There exist real numbers α_1, α_2 satisfying $\alpha_1+3b\alpha_2<1$ with α_1, $\alpha_2 \geq 0$ such that

$$S_b(Tx,Tx,Ty) \leq \alpha_1 S_b(x,x,y)+\alpha_2\max\{S_b(Tx,Tx,x),S_b(Tx,Tx,y),S_b(Ty,Ty,y),S_b(Ty,Ty,x)\},$$

for all $x,y \in X$. Then T has a unique fixed point x in X.

Proof. The proof follows easily by using the symmetry condition (2.1) instead of Lemma 3.1 in the proof of Theorem 3.6. □

Remark 3.8. *We note that Theorem 3.6 is a generalization of the Banach contraction principle on S_b-metric spaces. Indeed, if we take $\alpha_1<\frac{1}{b^2}$ and $\alpha_2=0$ in Theorem 3.6 we obtain the Banach contraction principle.*

Now we give an example of a self-mapping satisfying the conditions of Theorem 3.6 such that the condition of the Banach contraction principle is not satisfied.

Example 3.9. *We consider the S-metric space $(\mathbb{R},S)$ with*

$$S(x,y,z)=|x-z|+|x+z-2y|,$$

for all $x,y,z \in \mathbb{R}$ given in [19] and the self-mapping T of $\mathbb{R}$ as

$$Tx=\begin{cases} x+50 & \text{if } |x-1|=1 \\ 45 & \text{if } |x-1| \neq 1 \end{cases},$$

for all $x \in \mathbb{R}$ defined in [20]. Since every S-metric space is an S_b-metric space, $(\mathbb{R},S)$ is an S_b-metric space with $b=1$. Then the inequality (3.3) is satisfied for $\alpha_1=0$ and $\alpha_2=\frac{1}{5}$. Then T has a unique fixed point $x=45$ by Theorem 3.6. But T does not satisfy the condition of the Banach contraction principle since for $x=1$, $y=0$ we get

$$S(Tx,Tx,Ty)=10 \leq hS(x,x,y)=2h,$$

which is a contradiction with $h<1$.

4. An application of the Banach contraction to linear equations

In this section, we give an application of the Banach contraction principle on S_b-metric spaces to linear equations. To do this, we consider the S_b-metric space generated by

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|,$$

for all $x, y \in \mathbb{R}^n$. We note that the symmetry condition (2.1) is not necessary in the following example.

Example 4.1. *Let $X = \mathbb{R}^n$ be an S_b-metric space with the S_b-metric defined by*

$$S_1(x,y,z) = \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |y_i - z_i|,$$

for all $x, y, z \in \mathbb{R}^n$, where $b = 1$. If

$$\sum_{i=1}^{n} \left|a_{ij}\right| \leq h < 1, \ (1 \leq j \leq n)$$

then the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{4.1}$$

has a unique solution. Let T be defined by

$$Tx = Ax + b,$$

where $x, b \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Now we show that the self-mapping satisfies the contraction of the Banach contraction principle. For $x, y \in \mathbb{R}^n$ we get

$$\begin{aligned} S_1(Tx,Tx,Ty) &= 2\sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij}(x_j - y_j) \right| \leq 2\sum_{i=1}^{n}\sum_{j=1}^{n} \left|a_{ij}\right| \left|x_j - y_j\right| \\ &= 2\sum_{j=1}^{n}\sum_{i=1}^{n} \left|a_{ij}\right| \left|x_j - y_j\right| = \sum_{j=1}^{n} 2\left|x_j - y_j\right| \sum_{i=1}^{n} \left|a_{ij}\right| \\ &\leq hS_1(x,x,y). \end{aligned}$$

Then T satisfies the Banach contractive condition. Using Theorem 3.6, the linear equations system (4.1) has a unique solution.

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